Problem Set 4 due March 18, at 10 PM, on Gradescope

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue.

Problem 1:

Consider a matrix A such that the general solution to the equation:

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \qquad \text{is} \qquad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

for arbitrary numbers λ and μ.

(1) How many rows and columns does A have? *(5 points)*

(2) Based on the information in the equation above, what is the second column of A? *(5 points)*

(3) Find the entire matrix A. *(10 points)*

Problem 2:

An $m \times n$ matrix with rank 1 has the property that all its rows are multiples of each other, so they are of the form:

$$A = \begin{bmatrix} a_1 \boldsymbol{b}^T \\ \hline a_2 \boldsymbol{b}^T \\ \hline \cdots \\ \hline a_m \boldsymbol{b}^T \end{bmatrix}$$

for some non-zero <u>row</u> vector $\boldsymbol{b}^T = \begin{bmatrix} b_1 & \dots & b_n \end{bmatrix}$ and some scalars $a_1, ..., a_m$, not all 0.

(1) Write A as the product of a $m \times 1$ matrix with an $1 \times n$ matrix. *(5 points)*

(2) Use part (1) to obtain a simple formula for the symmetric matrix $A^T A$ in terms of the a_i's and b_j's? *Hint: if you're stuck, try the case when A is 3×2 for some intuition.* *(10 points)*

(3) What is the rank of $A^T A$ from part (2)? Explain. *(5 points)*

Problem 3:

Find a basis for the vector space spanned by the vectors:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 \\ 6 \\ -3 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ 3 \\ -5 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -5 \\ -8 \\ 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \\ 9 \\ -4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 3 \\ 8 \\ -5 \end{bmatrix}$$

Explain your method. *(20 points)*

Problem 4:

For any two subspaces $V, W \subset \mathbb{R}^m$, we will write $V + W$ for the subspace consisting of all vectors of the form $\boldsymbol{v} + \boldsymbol{w}$, for arbitrary $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in W$.

(1) If $\boldsymbol{v}_1, ..., \boldsymbol{v}_k$ is a basis of V and $\boldsymbol{w}_1, ..., \boldsymbol{w}_l$ is a basis of W, explain why the set $\boldsymbol{v}_1, ..., \boldsymbol{v}_k, \boldsymbol{w}_1, ..., \boldsymbol{w}_l$ spans the subspace $V + W$. *(5 points)*

(2) Show that the dimension of $V + W$ is $\leq$ the sum of dimensions of V and W. Give an example of when the inequality is strict (i.e. $\dim(V + W) < \dim V + \dim W$). *(10 points)*

(3) Given two matrices A and B of the same shape, what is the relation between $C(A+B)$ and the subspaces $C(A)$ and $C(B)$? *(5 points)*

(4) Use the previous parts to show that rank $(A+B) \leq$ rank $A +$ rank B. *(5 points)*

Problem 5:

Does there exist a ..., as in each of (1), (2), (3) below? (*If the answer is yes, show such a matrix and explain why it has the required property. If the answer is no, explain why such a matrix doesn't exist*).

(1) 2×3 matrix with column space spanned by $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and left nullspace spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$? *(5 points)*

(2) 3×2 matrix with row space spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and nullspace spanned by $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$? *(5 points)*

(3) 2×2 matrix with column space spanned by $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and left nullspace spanned by $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$?

(5 points)